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Special Theory of Relativity

for

B.Sc. (Final)

Subject: PHYSICS

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Special Theory of Relativity (STR): Study of motion of a body in the inertial frame of references. It was given by *Albert Einstein* in 1905.

Two postulates of Special Theory of Relativity:

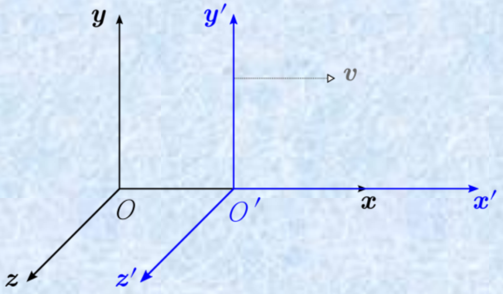
1. All laws of physics are same for the observers in all inertial frame of references
2. The speed of light in a vacuum ($c = 3\times10^8$ m/s) is the same for all observers, regardless of their relative motion or of the motion of the source of the light

Frame of reference: The well defined coordinate system with respect to which the motion of any object is defined.

Types of frame of references:

(i) Inertial or un-accelerated frame of reference: The frame of reference in which objects obey Newton's law of inertia and other laws of Newtonian mechanics.

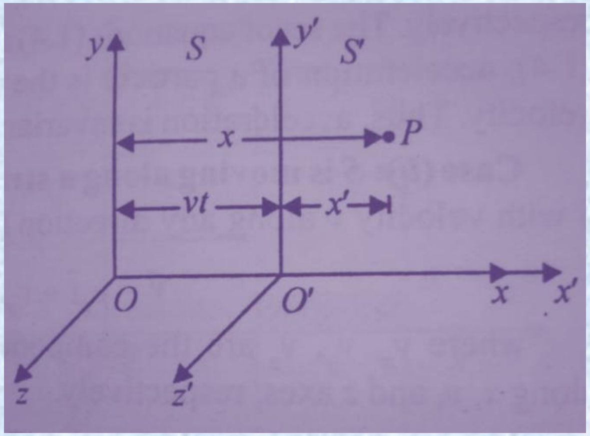
(ii) Non-inertial or accelerated frame of reference: The frame of reference in which objects do not obey Newton's laws.



This fig. Shows two inertial frame of references one is stationary and other is moving with some constant velocity *v*.

Galilean Transformation: Transforming the coordinates of one inertial frame to another.

Consider two inertial frame of references *S* and *S'* as shown in the figure. *S* is the coordinate system of (*x*, *y*, *z*, *t*) and *S'* is the coordinate system of *(x', y', z', t')*. Let the *S'* is moving with some velocity *v* with respect to (w.r.t.) *S* in +ve direction of *x*-axis.



In STR, we take coordinates in both space (*x*,*y*,*z*) and time *t*. Therefore position of any point will be taken as (*x*,*y*,*z*,*t*)

At time *t=0* both the frame of reference coincide.

If any event happens at point *P* at any time *t* as shown in fig. and observations are taken by both the observers sitting at point *O* and *O'* in *S* and *S'* frame of references. Then we have

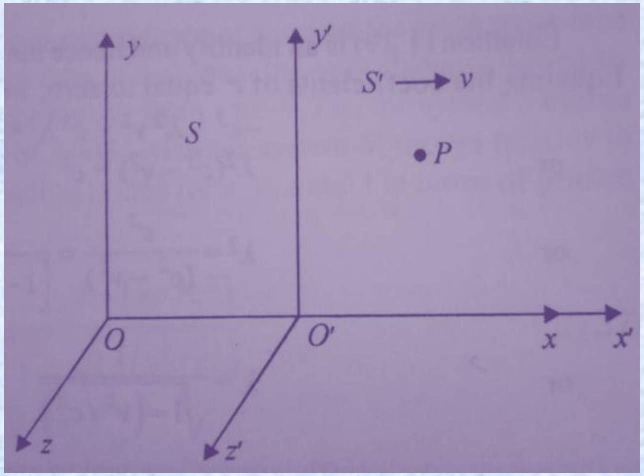
$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \quad (1)$$

Similarly, If we assume *S'* is at rest and *S* is moving with velocity *v* in negative direction of x-axis then

$$\left.\begin{aligned} x &= x' + vt \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}\right\} \quad (2)$$

NOTE: Set of equations (1) is called Direct Galilean Transformation and set of equations (2) is called Inverse Galilean Transformation

Lorentz Transformation: Galilean transformations are valid only for low speed *v* of moving frame of reference. Lorentz replaced the Galilean transformations and produce more accurate results for both low and high *v*.



Consider two inertial frames *S* and *S'*. *S'* is moving with constant velocity *v* w.r.t. *S* in +ve x-axis.

Lorentz assumed the relationship between *x* and *x'* as

$$x' = \lambda(x - vt) \qquad (1)$$

where, λ is any undetermined constant

According to First postulate of STR

$$x = \lambda(x' + vt') \qquad (2)$$

Moreover, *y'=y* and *z'=z*

From (1) and (2)

$$t' = \lambda t + \left(\frac{1-\lambda^2}{\lambda v}\right) x \qquad (3)$$

Consider a light signal produced at *t=t'=0* when the origins of two frames coincide. This signal propagates as time passes.

Now, apply second postulate of STR (The velocity of light *c* is same in *S* and *S'*)

After time *t* the distance covered by light pulse as observed in *S* $\quad x = ct \qquad (4)$

After time *t'* the distance covered by light pulse as observed in *S'* $\quad x' = ct' \qquad (5)$

Substitute the values of *x'* and *t'* from equations (5) and (3) into equation (2) and after solving we get

$$x = ct\left[\frac{1+\frac{v}{c}}{1-\left(\frac{1}{\lambda^2}-1\right)\frac{c}{v}}\right] \qquad (6)$$

Equation (6) must be same as equation (4). On comparing them → $$\left[\frac{1+\frac{v}{c}}{1-\left(\frac{1}{\lambda^2}-1\right)\frac{c}{v}}\right]=1 \qquad (7)$$

And we get $$\lambda = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (8)$$

Putting the value of λ in equation (1) we get $$x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (9)$$

Similarly, on putting the value of *x* from eq^n (6) and λ from eq^n (8) into eq^n (3) we get

$$t' = \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (10)$$

So Lorentz transformation becomes

$$x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} \qquad y' = y$$

$$z' = z \qquad t' = \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Similarly, if *S* is moving with velocity *v* w.r.t. *S'* in opposite direction (-ve x-axis) then we get

$$x = \frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}} \qquad y = y'$$

$$z = z' \qquad t = \frac{t'+\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

→ Inverse Lorentz Transformation

Velocity Addition Theorem

The Lorentz transformation equations lead to a relativistic formula for the addition of velocities which is more accurate as compared to classical mechanics.

Consider two frames of references S and S'. Let the frame S' is moving with a velocity *v w.r.t.* to S along the positive direction of X-axis. Let a body also move along the positive direction of X-axis. If the body moves a distance *dx* in time *dt* in frame S then its velocity in frame S is given by

$$u = \frac{dx}{dt} \qquad (1)$$

In frame S', the distance moved by the body will appear as dx' in time dt'. Hence the velocity of the body will appear as

$$u' = \frac{dx'}{dt'} \qquad (2)$$

Now from Lorentz transformation equations, we have

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad (3) \qquad \text{and} \qquad t' = \frac{t - vx/c^2}{\sqrt{1 - vx/c^2}} \qquad (4)$$

Differentiating eq^n (3) and (4), we get

$$dx' = \frac{dx - v.dt}{\sqrt{1 - v^2/c^2}} \qquad (5) \qquad \text{and} \qquad dt' = \frac{dt - \frac{v.dx}{c^2}}{\sqrt{1 - v^2/c^2}} \qquad (6)$$

From eq^n (5) and (6), the velocity of the body in frame S' is given by

$$u' = \frac{dx'}{dt'} = \frac{u - v}{1 - \frac{uv}{c^2}} \qquad (7)$$

When v << c, $u' = u - v$ (classical mechanics formula)

The inverse velocity transformation equation can be obtained by replacing *v* by –*v* and interchanging primed and unprimed coordinates. Thus we have

$$u = \frac{dx}{dt} = \frac{u' + v}{1 + \frac{u' v}{c^2}} \qquad (8)$$

If $u' = c$, i.e. If the moving body along positive direction of X-axis be photon then its velocity relative to frame S will be

$$u = \frac{dx}{dt} = \frac{c + v}{1 + \frac{v}{c}} = c \qquad (9)$$

Thus, the speed of light is same in all inertial frames of references.

Length Contraction: In Newtonian mechanics, length of an object is same for all observers but according to STR length is a variable physical quantity and it contracts when the object moves.

Consider a rod of length L_0 placed in a moving frame of reference *S'* . The observer *O'* observers the end coordinates of the rod as x_1' and x_2'.

Therefore, $$x_2' - x_1' = L_0 \qquad (1)$$

And according to the observer *O* sitting in stationary frame of reference *S*, the coordinates of ends of the rod are x_1 and x_2 . Hence the observed length of the rod by the observer *O* is

$$L = x_2 - x_1 \qquad (2)$$

Now, According to Lorentz transformation $x_1' = \frac{x_1 - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$ and $x_2' = \frac{x_2 - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$

Put the value of x_1' and x_2' in eq^n (1), we get

$$L_0 = \frac{x_2 - x_1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (3)$$

or, $$\boxed{L = L_0 \sqrt{1 - \frac{v^2}{c^2}}} \qquad (4) \qquad \because \quad x_2 - x_1 = L$$

NOTE: From eq^n (4), the observed length of the rod decreases as *v* increases.
Case 1: when (v/c) <<1 then $L=L_0$ (Length does not appear to be increased)
Case 2: when (v=c) then L=0 (Length of the object contracts to a point)

Time Dilation: Time dilation is the lengthening of the time interval between two events for an observer in an inertial frame that is moving with respect to the rest frame of reference.

Consider two inertial frames *S* and *S'*. *S'* is moving with respect to *S* in +ve direction of x-axis with a velocity *v*. Imagine a gun placed at the fixed position *(x',y',z')* in the frame *S'*. Suppose it fires two shots at time t_1' and t_2' and observed by the observer *O'* sitting in *S'*.

According to observer *O'* the time interval between the shots is given by

$$t_2' - t_1' = t_0 \quad \text{(say)} \qquad (1)$$

Now, according to observer *O* sitting in *S* the observed time interval between the shots is given by

$$t_2 - t_1 = t \quad \text{(say)} \qquad (2)$$

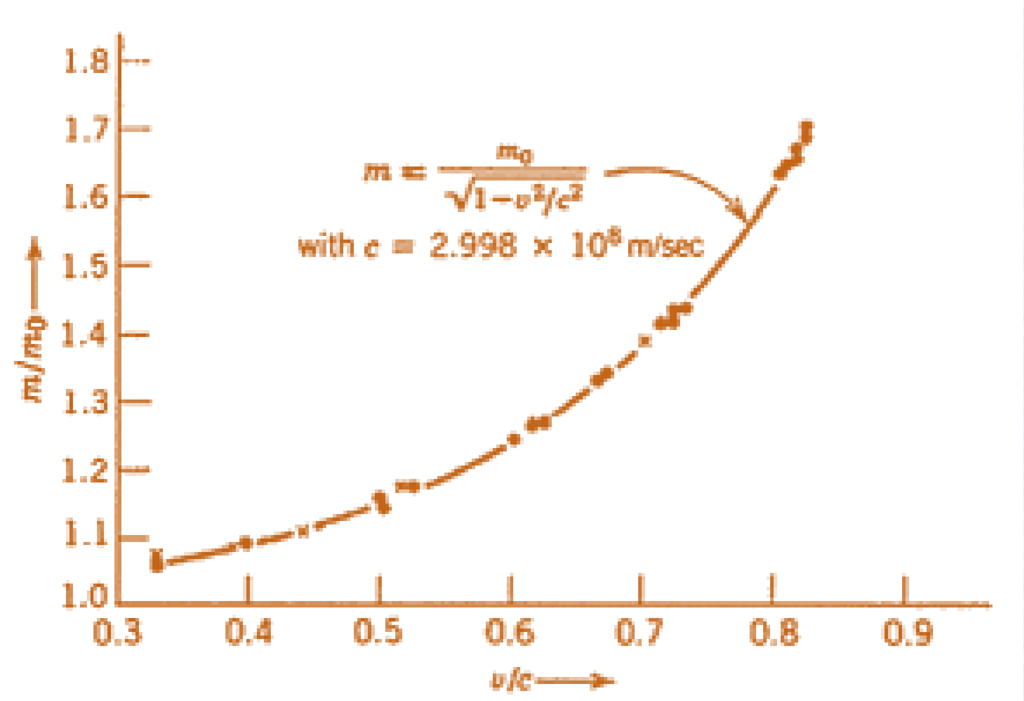
From inverse Lorentz transformation equations, we have $t_1 = \dfrac{t_1' + \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$ (3) and $t_2 = \dfrac{t_2' + \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$ (4)

On putting the values of t_1 and t_2 in eq^n (2), we have

$$t = \frac{t_2' - t_1'}{\sqrt{1-\frac{v^2}{c^2}}} \quad \Rightarrow \quad t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (5)$$

NOTE: Eq^n (5) shows that t>t_0 i.e. the time interval appears to be lengthened by a factor $\frac{1}{\sqrt{1-v^2/c^2}}$ which is observed by the observer in *S* frame. This is known as time dilation.

Variation of mass with velocity: In Newtonian physics, mass of a particle is considered to be a constant quantity. However, according to relativity the mass is also a variable quantity just like length and time.



The mass of a moving particle is given by

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (1)$$

where,
m_0 =the rest mass of the particle (mass of the particle when it is at rest)
v=velocity of the particle
c= speed of light

NOTE: Mass of the particle increases as the velocity increases

Einstein's Mass Energy Relation

According to STR, the mass of the moving particle is given by $$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (1)$$

If a force F is applied to the particle then it will be equal to the rate of change of momentum (mv). Therefore, $$F = \frac{d(mv)}{dt} \quad (2)$$

If the particle is displaced a distance dx by this force F, the work done $F.dx$ is stored as the kinetic energy (dE_k). Hence $$dW = dE_k = F.dx \quad (3)$$

From eq^n (2) and (3), we have $$dE_k = \frac{d(mv)}{dt}.dx = \frac{dx}{dt} d(mv) \quad (4)$$

or, $dE_k = v\, d(mv) = v(mdv + vdm)$ $\quad \because \frac{dx}{dt} = v$ $\quad \Rightarrow \quad$ $dE_k = (mv\, dv + v^2\, dm) \quad (5)$

Using eq^n (1), we have $$m^2c^2 - m^2v^2 = m_0{}^2c^2 \quad (6)$$

Differentiating eq^n (6) both side, we get

$$2m\, dm\, c^2 - 2m\, dm\, v^2 - 2\, v\, dv\, m^2 = 0 \quad \Rightarrow \quad c^2 dm = v^2 dm + mv\, dv \quad (8)$$

From eq^n (5) and (8), we have $$dE_k = c^2 dm \quad (9)$$

Integrate both sides $\Rightarrow$ $$\int_0^{E_k} dE_k = c^2 \int_{m_0}^{m} dm \quad \Rightarrow \quad E_k = mc^2 - m_0c^2 \quad (10)$$

Cont...

$$\because \quad E = E_k + m_0 c^2 \qquad (11)$$

Total energy of the moving particle (E)

Kinetic energy of the particle (E_k)

Rest mass energy of the particle ($m_0 c^2$)

From eq^n (10) and (11), we have

$$\because \quad E = mc^2 \qquad (12)$$

Above eq^n (12) is called mass energy relation

Inference: Energy of the particle can be converted into mass and vice-versa.

Numerical Problems

1. Show that under Lorentz transformation $x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$

2. With what velocity a particle should move so that its mass appears to increase by 20% of its rest mass?

3. If the kinetic energy of a body is doubled of its rest mass energy, calculate its velocity.

4. What is the length of a metre stick moving parallel to its length when its mass is 3/2 times of its rest mass?

5. Half life of a particle is 17.8 nanosecond. What will be the half life when its speed is 0.8 c ?

6. With what velocity should a rocket move so that every year spent on it corresponds to 4 years on earth?

7. A space-ship moving away from earth with velocity of 0.5 c fires a rocket whose velocity relative to spaceship is 0.8 c, (a) away from earth, (b) towards the earth. What will be the velocity of the rocket as observed from the earth in the two cases?

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3. Dr. J. P. Agarwal, Relativity and Statistical Physics, Pragati Prakashan, Meerut (2019).